

BOUNDS ON THE NUMBER OF SMALL DISTANCES IN A FINITE PLANAR SET

K. VESZTERGOMBI

To Professor László Fejős Tóth on his 70th birthday

Abstract

Let m_j denote the number of times the j^{th} smallest distance occurs among a set S of m points in $\mathbb{R}^2$. We show that $m_j \leq 3jm$ and $m_2 \leq 5m$ and $m_1 + m_2 \leq 6m$. We give a construction with $m_2 \sim \frac{24}{7}m$.

Introduction

Let S be a set of m points in $\mathbb{R}^2$. We denote by t_1 the smallest distance between two points, by t_j the j^{th} smallest distance and by $m_1, \dots, m_j, \dots$ the number of occurrences of the distance $t_1, \dots, t_j, \dots$ resp. In [1] Harborth gave an exact upper bound for m_1 . In the recent paper we give an upper bound on m_j , namely $m_j \leq 3jm$, and a sharper bound on m_2 ; namely $m_2 \leq 5m$. We give a construction in which $m_2 \sim \frac{24}{7}m$. In the last part we prove $m_1 + m_2 \leq 6m$, for which the lattice of regular triangles is sharp.

1

Let us assign a graph G_j to the case of the j^{th} distance: the vertices of the graph correspond to the points of the set S , and two vertices are connected by an edge if the distance of the corresponding points equals t_j . Let $d_j(v)$ denote the degree of $v \in S$ in G_j . We get the upper bound for m_j , by giving bounds for the degrees in G_j .

LEMMA 1. For all $v \in S$, $d_j(v) \leq 6j$.

PROOF. Suppose a vertex v has at least $6j+1$ neighbours in G_j . Then if we take the circle of radius t_j around v , then there must exist an arc of angle $\leq \frac{\pi}{3}$ on the circle which contains at least $j+2$ points (including the endpoints of the arc). Then this means that there are at least j different distances smaller than t_j , between the points on the arc, which contradicts the assumption that t_j is the j^{th}

1980 *Mathematics Subject Classification*. Primary 51D20, 52A37; Secondary 05B30.

Key words and phrases. Point set, distances, second smallest distances between points.

smallest distance. Furthermore, v can have $6j$ neighbours in G_j if the neighbours form a regular $6j$ -gon on the circle of radius t_j around v .

From this lemma we get the upper bound:

PROPOSITION.

$$m_j \leq 3jm.$$

In the case $j=1$ if we take the lattice of regular triangles we get the upper bound asymptotically in the sense that we do not count the deficiency of the boundary points of the set.

2

In this part we prove a sharper upper bound on m_2 . By Lemma 1 we know that the degree of any point in G_2 is at most 12. But one can notice that the neighbours of a point of degree 12 must have essentially smaller degree. We call the sum of the degrees of the endpoints of an edge the *degree of the edge*. We prove for the degrees of the edges in G_2 the following upper bound:

LEMMA 2. *The degree of any edge of G_2 is at most 20.*

PROOF. Let uv be any edge of G_2 ; suppose that $d_2(v) \geq d_2(u)$. If $d_2(v) \leq 10$ then $d_2(u) + d_2(v) \leq 20$, trivially. So there are two cases to consider:

(a) Case $d_2(v) = 12$.

We want to show that the other endpoint u has degree at most 8. By Lemma 1 the neighbours of v in G_2 form a regular 12-gon. Let us denote them by u_i for $1 \leq i \leq 12$, ($u_1 = u$) and the circle around v of radius t_2 by C_1 . The smallest distance t_1 must be equal to the side of this regular 12-gon, so $t_1 = d(u_i, u_{i+1})$, obviously. Now let us consider u_1 . Then u_1 already has some neighbours of distance t_2 ; namely u_3 , u_{11} and evidently v . We draw the circle C_2 around u_1 of radius t_2 . Let us denote by r_i the vertices of the regular 12-gon on C_2 (some of them coincide with earlier mentioned points; see Fig. 1). Then on the arc vu_3 (of angle $\frac{\pi}{3}$) there cannot be

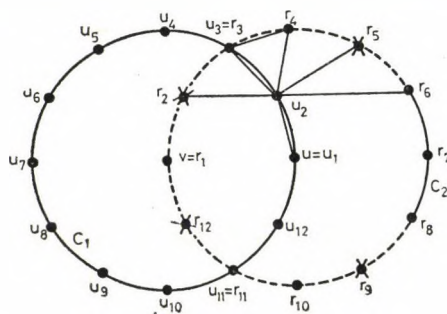


Fig. 1

any other point than r_2 because $d(v, r_2) = t_1$ and $d(v, u_3) = t_2$. But r_2 cannot belong to the set S because $t_1 < d(r_2, u_2) < t_2$, obviously. Similarly on the arc vu_{11} (of angle $\frac{\pi}{3}$) there cannot be any point of the set S . There cannot be any point except r_4 on the arc u_3r_4 (of angle $\frac{\pi}{3}$) because $d(u_3, r_4) = t_1$, and again on the arc r_4r_6 (of angle $\frac{\pi}{6}$) because it would give distance between t_1 and t_2 . But r_6 is forbidden because $t_1 < d(u_2, r_6) < t_2$. There cannot be any point s , $s \neq r_6$ on the arc r_6r_8 because $t_1 < d(u_2, s) < d(u_2, r_6) = t_2$. This argument goes through symmetrically for the arc vr_8 (of angle $\frac{5\pi}{6}$). So u_1 has at most 8 neighbours of distance t_2 :

$$v, u_3 = r_3, r_4, r_6, r_7, r_8, r_{10}, r_{11} = u_{11}.$$

(b) Case $d_2(v) = 11$.

(i) First we deal with the case when all the neighbours u_i ($1 \leq i \leq 11$) of v in G_2 form a regular 11-gon. Let us call C_1 the circle of radius t_2 around v . Obviously, $t_1 = d(u_i, u_{i+1})$. Let us consider the neighbours of u_1 of distance t_2 . We denote by C_2 the circle of radius t_2 around u_1 . Let us denote by r_i ($1 \leq i \leq 11$) the vertices of the regular 11-gon on C_2 where $r_1 = v$ (as on Fig. 2). On the arc vr_2 (of angle $\frac{2\pi}{11}$)

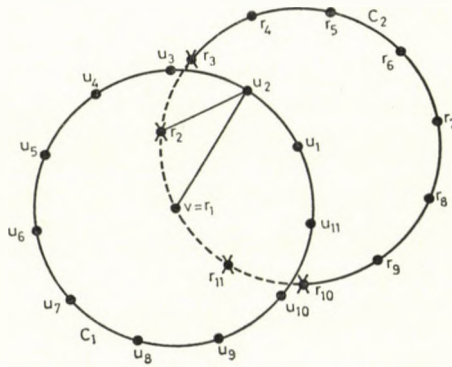


Fig. 2

there cannot be any point $s \in S$ at distance t_2 from u_1 because $d(v, s) < t_1$ would hold. r_2 is forbidden because

$$t_1 = d(v, r_2) < d(r_2, u_2) < d(v, u_2) = t_2,$$

because for the corresponding angles of the triangle we have the same inequalities

$$\frac{2\pi}{11} < \frac{5\pi}{22} < \frac{13\pi}{22}.$$

The next possibility would be the crossing point of the circles C_1, C_2 but it is an inner point of the arc u_2u_3 so it would give a smaller distance than t_1 . Symmetrically the arc vr_{10} (of angle $2 \cdot \frac{2\pi}{11}$) is forbidden. So what remained from C_2 is an arc of angle $< \frac{14\pi}{11}$. On this arc the set S can have less than 9 points, since there cannot be a smaller angle than $\frac{2\pi}{11}$, which belongs to t_1 .

(ii) Secondly, we deal with the case when the u_i 's do not form a regular 11-gon. Now there exist u_i, u_{i+1} (the i 's are understood mod 11) for which $d(u_i, u_{i+1}) > t_1$, but then it has to be at least t_2 . On the other hand for all u_i, u_{i+2} pairs $d(u_i, u_{i+2}) \equiv t_2$ must hold which means that the angle belonging to t_1 must be at least $\frac{\pi}{6}$. From this it follows that the only other possibility for having 11 neighbours of v in G_2 is that the u_i 's form a regular 12-gon with one point deleted. Now those u_i 's which are "far" from the missing point can have ≤ 8 neighbours by the same argument as in Case (a). So this kind of edges vu_i have degree ≤ 20 .

We still have to check the u_i 's which are "close" to the missing point, as indicated on Fig. 3.a, b. In the case when u_2 is the missing neighbour of v (see Fig. 3a) a similar argument as for $d_2(v)=12$ shows that we have to consider on the arc vr_4 (of angle $\frac{\pi}{2}$) the point r_2 . But $t_1 < d(r_2, u_5) < t_2$ which means that $d_2(u_1) \leq 9$ because r_9 and r_{12} are naturally forbidden.

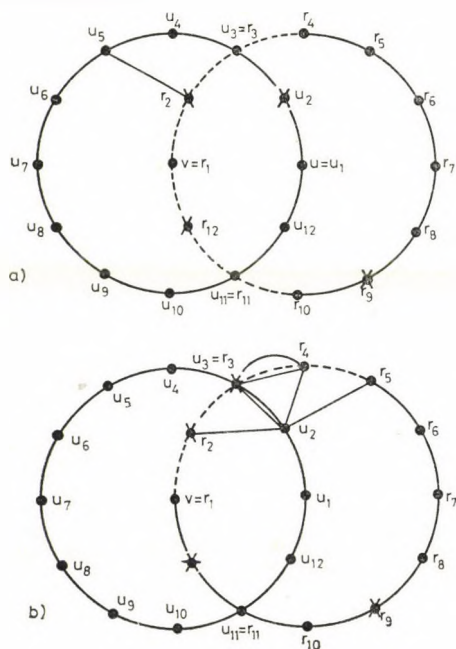


Fig. 3

In the case when u_3 is the missing neighbour of v (see Fig. 3b) then again the first possible neighbour of u_1 would be r_2 , but $t_1 < d(r_2, u_2) < t_2$, so r_2 is forbidden. Since the triangle $u_2 r_4 u_3$ is regular, on the arc $u_3 r_4$ we cannot have any point s because $d(u_2, s) < t_1$ would hold. So this shows $d_2(u_1) \leq 8$. From this argument one can see that if any other u_i is missing the existing points forbid enough points to have the inequality for the degree of the edges vu_i .

THEOREM. $m_2 \leq 5m$.

PROOF. From Lemma 2 we know that for any edge e of G_2 $d(e) \leq 20$. Now if we sum for all edges e_i , $1 \leq i \leq m_2$ of G_2 we know the following:

$$d(e_1) + \dots + d(e_{m_2}) \leq 20m_2.$$

On the other hand we know:

$$\sum_{i=1}^{m_2} d(e_i) = \sum_{j=1}^m [d(v_j)]^2.$$

By the inequality for the algebraic and the quadratic mean for the degrees of the points:

$$\frac{2m_2}{m} = \frac{d(v_1) + \dots + d(v_m)}{m} \leq \sqrt{\frac{[d(v_1)]^2 + \dots + [d(v_m)]^2}{m}} \leq \sqrt{\frac{20m_2}{m}}.$$

From these we get the desired inequality.

3

We give a construction which gives more second smallest distances than the regular triangular lattice, but we do not know what is the best construction because what we get is far from the bound proved above. We take the regular hexagonal lattice. On every edge of the hexagons we take two more points so that in every hexagon these new points form a regular 12-gon (see Fig. 4). The set S consists of the centers c_i of these 12-gons and the vertices v_i of the 12-gons. One can easily check that the edge of the 12-gon is the smallest distance in the set. The second

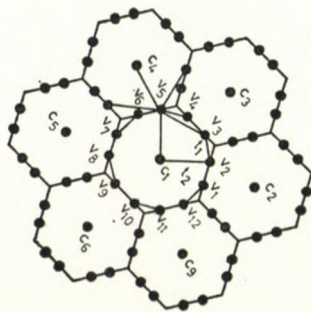


Fig. 4

smallest distance t_2 equals the radius of the circumscribed circle of the 12-gons. For the centers c_i we have $d_2(c_i)=12$, and for the other type of points v_i , $d_2(v_i)=6$, except for those points on the boundary of the figure. So in this construction:

$$m_2 = 3\frac{3}{7}m + o(m).$$

4

Next we give an upper bound for the number of edges of $G_1 \cup G_2$. The following theorem was suggested by P. Erdős:

THEOREM. $m_1 + m_2 \leq 6m$.

This theorem follows immediately from the next lemma:

LEMMA 3. *For every vertex v , $d_1(v) + d_2(v) \leq 12$.*

PROOF. Suppose $d_1(v) + d_2(v) \geq 13$, then there exist two neighbours s_1, s_2 of v in $G_1 \cup G_2$ such that $\angle s_1 v s_2 < 30^\circ$. Therefore $d(s_1, v) = d(s_2, v) = t_1$ is impossible, from this it follows that $t_2 > \sqrt{3}t_1$. By Lemma 1 we know that $d_1(v) \leq 6$. If $d_1(v) \geq 3$ one can easily check that $t_2 \leq \sqrt{3}t_1$. So this contradicts the above inequality. So we may suppose $d_1(v) \leq 2$. Then we have to consider the following two cases:

Case (a). $d_2(v)=12$.

We saw in Lemma 1 that in this case the neighbours u_i of v in G_2 form a regular 12-gon. Obviously, $d(u_i, u_{i+1}) = t_1$. One can easily check that the radius of the circumscribed circle of the triangle vu_1u_2 equals t_1 . Say s is the center of this circle (see Fig. 5). So any other point of the circle C in the triangle vu_1u_2 around v with radius t_1 is either from u_1 or from u_2 closer than t_1 . So s is the only possibility in the triangle vu_1u_2 for having distance t_1 from v . On the other hand one can easily check

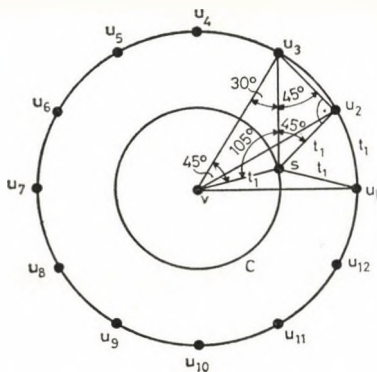


Fig. 5

that $d(u_3, s) = \sqrt{2}t_1$. So $t_1 < d(u_3, s) < t_2$, which means that s is forbidden. So if $d_2(v) = 12$ then $d_1(v) = 0$.

Case (b). $d_2(v) = 11$.

(i) We have seen in Case (a) if $d_2(v) = 12$, then $d_1(v) = 0$. If the 11 neighbours of v in G_2 form a regular 12-gon with one point missing there cannot be point on the circle of radius t_1 around v except inside an arc of angle 60° , but this allows only one point, so $d_1(v) \leq 1$.

(ii) If the neighbours u_i of v in G_2 do not form a regular 12-gon with one point missing, then the u_i 's form a regular 11-gon, by Lemma 2 (b). Let us take the circle C_1 around v of radius t_1 . If r is a point on C_1 in the triangle vu_1u_2 then $d(u_1, r) < t_2$ and $d(u_2, r) < t_2$ (see Fig. 6). Furthermore $d(u_1, r) = d(u_2, r) = t_1$ is possible only for the regular 12-gon. So this means that in this case $d_1(v) = 0$. So we finished the proof of our statement.

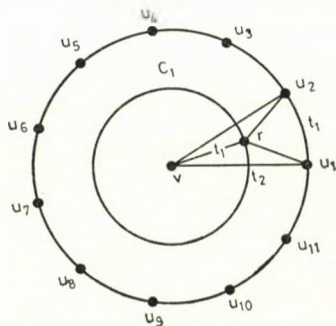


Fig. 6

REMARK. If we take the regular triangular lattice we get for all points v in the set S $d_1(v) + d_2(v) = 12$, except for those points on the boundary of the figure.

REFERENCE

[1] HARBORTH, H. (unpublished).

(Received May 20, 1985)

EÖTVÖS LORÁND TUDOMÁNYEGYETEM
TERMÉSZETTUDOMÁNYI KAR
ANALÍZIS TANSZÉK
MŰZEUM KRT. 6-8
H-1088 BUDAPEST
HUNGARY